Indian Statistical Institute, Bangalore

B. Math. III Second Semester

Differential Geometry II: Final Exam (Back paper)

Duration: 3 hours **Date : May 29, 2015**

Maximum Marks: 50

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(1) *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a C^∞-smooth function. Let $q \in \mathbb{R}$ be a regular value for f such that $S = f^{-1}\{q\} \neq \emptyset$.*

i) *Prove that S is a smooth manifold of dimension $n-1$*

ii) *Let $p \in S$. Show that the tangent space $T_pS = KerDf(p)$.*

iii) *Calculate T_pS^n, where $S^n \subset \mathbb{R}^{n+1}$ is the standard unit sphere.*

iv) *Show that the tangent bundle of the sphere $S^n \subset \mathbb{R}^{n+1}$ is the set*

$$\{(x,v) \in S^n \times \mathbb{R}^{n+1} :< v, x >= 0\}.$$

where $< .,. >$ is the standard inner product in $\mathbb{R}^{n+1}$.

(20 marks)

(2) *Define a function $F : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ by*

$$F(x,y) =< x, y >,$$

where $< .,. >$ is the standard inner product in $\mathbb{R}^n$

i) *Find $DF(a,b)$*

ii) *If $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ is differentiable and $|f(t)| = 1$ for all $t \in \mathbb{R}$. Show that $< (f'(t))^T, f(t) >= 0$ for all $t \in \mathbb{R}$.*

(10 marks)

(3) *Let U be an open subset of $\mathbb{R}^n$. $f_i \in C^\infty(U)$, $i = 1, \ldots, n$. Show that*

$$df_1 \wedge \cdots \wedge df_n = \det\left[\frac{\partial f_i}{\partial x^j} dx^1 \wedge \cdots \wedge dx^n\right],$$

where $x^1, \ldots, x^n$ are the coordinates of $\mathbb{R}^n$.

(10 marks)

(4) i) *Calculate the Riemannian metric of S^2 induced from the standard Riemannian metric of $\mathbb{R}^3$.*

ii) *What is the connection in $\mathbb{R}^3$ with respect to the standard Riemannian metric $< .,. >$?*

iii) *Calculate the Levi-Civita connection of S^2 with above mentioned Riemannian metric.*

iv) *Find the Levi-Civita connection of any surface S in $\mathbb{R}^3$, where the Riemannian metric on the surface is induced by the standard Riemannian metric in $\mathbb{R}^3$.*

(24 marks)

Note: You can use well-known theorems taught in the class, but you need to write precise statement of the theorem you are using.